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Working with Logarithms: Students' Misconceptions and Errors

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Abstract: This study examines secondary school students' understandings and misconceptions when working with logarithms using a specially designed test instrument administered to 81 students in two Singapore schools. Questions were classified by cognitive level. The data were analysed to uncover the kinds of errors made and their possible causes. Students appear capable of doing routine calculations but less capable when answering questions which require higher levels of cognitive thinking. In addition, many errors are not due to lack of knowledge but appear to be based on over-generalisation of algebraic rules. Suggestions for practice based on these findings are provided.

Introduction

Anecdotal evidence from teachers and colleagues over the years has consistently confirmed that teaching logarithms in secondary school is difficult; and, even though students can often "do" the questions that are in the text and on the examinations, their understanding of the fundamental nature of logarithms remains in doubt. Indeed, when some of these students decide to pursue a teaching career and come for pre-service teacher training, we, as teacher educators, often find that they do not have the grasp of the conceptual underpinnings of logarithms that we would like, even though many have impressive mathematics examination grades.

When logarithms were first developed in the seventeenth century, they revolutionised astronomy because of the way they permitted previously impossible calculations to be done. Despite the fact that computers have taken over this computational role, logarithms are still tremendously important to the field of mathematics and science. For example, logarithms are applied when studying population growth and determining the *pH* value of a solution. Consequently, logarithms can still be found in secondary mathematics curricula around the world. Their importance, combined with the perceived difficulty, motivated this research project – a project designed to find out more about the ways in which students understand logarithms and the kinds of errors and misunderstandings that are evident.

This paper reports the results of that research. Although the study was undertaken in Singapore, the results were consistent with our own classroom experiences in different countries. The paper begins with a review of the current research findings in this area and follows with a discussion of the method, instrumentation and results. The implications of the findings for classroom practice are presented as a way of helping teachers make use of this research in a meaningful way.

Conceptual Framework

As seasoned secondary school teachers familiar with how students process and develop mathematics concepts, our experiences provided a fundamental framework for our thinking about learning and teaching logarithms. There appears to be very little research literature available to either confirm or deny the validity of our experiences. There are, however, numerous studies focused on students' learning difficulties in elementary algebra (see, for example, Booth, 1983; Milton, 1988; Mestre, 1989; Booth & Watson, 1990; Loh, 1991; Parish & Ludwig, 1994; Ng, 1996; Schwartzman, 1996; Ong, 2000; Ng, 2002). It thus seemed useful to frame the topic of logarithms within algebra itself so that these findings can be used to improve students' learning of this particular topic.

Although algebra is often thought of by students as merely strings of letters or a set of tools for manipulating them, it can be considered from four distinct perspectives: generalised arithmetic; a study of procedures; a study of relationships among quantities; and, a study of structure (Usiskin, 1988). We have chosen to study the topic of logarithms from these four perspectives.

If algebra is considered to be generalised arithmetic, then variables can be used to generalise patterns so that $3+5=5+3$ can be generalised as $a+b=b+a$. In this case, the letters do not represent unknowns or variables but can be thought of as symbolic place holders. In the context of logarithms, the pattern $\log_2 3+\log_2 5=\log_2 15$, $\log_2 3+\log_2 6=\log_2 18$ and $\log_2 3+\log_2 7=\log_2 21$ can be translated and generalised as $\log_a x+\log_a y=\log_a xy$ in which the variables do not represent unknowns but rather are used to provide a generalised pattern.

When viewing algebra as a study of procedures for solving certain kinds of problems, variables can be thought of as unknowns. In the study of logarithms, questions do not usually require the translation of a word problem into a logarithmic equation, as is often the case in elementary algebra. However, the translation may be articulated when a teacher explains the structure of the equation. For example, to solve the equation $\log_2(x+5)=3$, the teacher may explain the equation as "3 is the power to which 2 needs to be raised to give 5 more than a certain number. What is this number?" The equation is then solved with an appropriate procedure, making use of students' understanding of the meaning of logarithmic and exponential forms. A student might first convert the logarithmic equation into an exponential equation:

$x+5=2^3$ and then subtract 5 on each side, arriving at $x = 3$. The "certain number" is 3 and the result is easily checked. In this regard, x in the equation $\log_2(x+5)=3$ is viewed as an unknown.

Thinking of algebra as the study of relationships among quantities where a variable can be used as either an argument (the domain value of a function) or a parameter (a number on which other numbers depend) yields yet another perspective. In other words, the variables become quantities in which a change in one variable determines a change in the other. For example, in the area formula for a rectangle $A=L\times B$, where A, L and B represent the area, length and breadth respectively, a change in B produces a corresponding change in A when L remains a constant. Since none of the letters in the formula is being evaluated, the letters representing the various quantities do not display the true nature of an unknown. Neither does the formula show the character of generalisation such as $a + b = b + a$. The essence of this conception is evidenced when answering the question, "*What happens to the value of A as B becomes larger and larger, given L = 10?*" Since the area formula $A=L\times B$ shows how the dimensions of a rectangle and its area are related, it would seem that both L and B are parameters while A is an unknown. But the status of L is changed from a parameter to a constant when the length 10 is substituted into the formula. The letter L then represents a constant and not a parameter, leaving only the parameter B because B does not represent any particular number. The question does not ask for a value of A, thus disqualifying it as an unknown. Hence A is dependent on B, and is thus treated as an argument.

Logarithms can be viewed as a study of relationships among quantities as well. While discussing the constraints on x in the relationship $y=\log_2 x$, students can be asked to explore the value of y as x approaches zero. Here, there is no generalisation of a pattern, so neither x nor y is used to represent a generalised pattern. Nothing specific is being solved for, thus eliminating the possibility that y is an unknown. Rather x and y should be treated as the argument and the parameter respectively in this case.

Although the study of algebra as the study of structures, such as groups or vector spaces where variables are treated as arbitrary objects related by certain algebraic rules, is not common in secondary school, it is important in higher mathematics. Many logarithmic questions can be considered from this perspective. For instance, if students are asked to "Simplify $\log_a 8-2\log_a 4$" and "Evaluate $\log_b\sqrt{b}$", the expressions do not represent any pattern to be generalised, any equation to be solved nor any function. The variables a and b are simply arbitrary objects which are manipulated according to some operational set of rules that have been established in a logically consistent manner.

Thus the role of algebra in the learning of logarithms is crucial and cannot be neglected if successful mastery of logarithms is to take place. In fact, treating an algebraic expression as an object, rather than as a process, has been widely advocated by many researchers (Mason, 1996). One advantage of this approach is that when students regard the pattern as an object, they perceive the pattern as a structure (Sfard, 1991). Once students acquire such abilities to generalise and to perceive patterns as structures, they can progress further to develop more complex algebraic skills such as those needed in the learning of logarithms. For example, when simplifying the expression $2\log_a 5 - 3\log_a 2$ to a single logarithm, students need to treat both the terms $2\log_a 5$ and $3\log_a 2$ as objects which can be replaced by $\log_a 25$ and $\log_a 8$ respectively before applying the quotient law $\log_a x - \log_a y = \log_a \frac{x}{y}$ to give the answer of $\log_a \frac{25}{8}$.

The incorrect perception of mathematical ideas as objects can also lead to problems. For example, students often see the notation for logarithm "log" as an object rather than as an operation. Yen (1999), in his analysis of the types of errors Australian students made in the 1998 High School Certificate (HSC) Mathematics examination, showed that some students divided both sides of the equation $\ln(7x-12) = 2\ln x$ by "ln" as if it were a variable to obtain $7x - 12 = 2x$ when solving the equation. A study conducted by Kaur and Boey (1994) in a Junior College in Singapore found that not all students realised that the simplification of $\frac{\log 16 - \log 8 + \log 4}{\log 3}$ to $\frac{\log(16-8+4)}{\log 3}$ was incorrect. It appears the root of the misconception is the mistaken notion that the "log" in the expression, $\log x + \log y$, is a common factor. This error is actually quite common and is often called the linear extrapolation error (Matz, 1980). For example, when asked to solve the equation $\ln(7x-12) = 2\ln x$, students claim that $\ln(7x-12) = \ln 7x - \ln 12$, clearly treating "ln" as a variable and distributing it over $7x$ and 12 (Yen, 1999).

This error is not the only potential problem when working with logarithms, however. Errors can also be caused by the students' own formulation of rules that work well for some questions but not in general. Lee and Heyworth (1999) reported that when a student was asked to simplify the expression $\log 60 - \log 6$, she responded with the answer $\frac{\log 60}{\log 6}$. An interview with this student showed that she had established the rule that "you can always change subtraction to division when doing logarithms", which she had used successfully in many cases. A similar example was noted by Kaur and Boey (1994) where a few students were not aware that $\frac{\log 12}{\log 3} = \log \frac{12}{3}$ is an incorrect statement. The source of such misinterpretations is unclear. Lopez-Real (2002) illustrated an incident in which the teacher used the phrase "if the logarithms of two numbers are the same, then these two numbers are the same" as a justification to explain the elimination of the "log" symbol in the expressions $\log x = \log 2 \Rightarrow x = 2$ and $\log(3x-10) = \log(6-x) \Rightarrow 3x - 10 = 6 - x$. Students, however, constructed their own understanding of this idea and used it to simplify the expression $\frac{\log a^3b^2 - \log ab^2}{6\log a}$ to $\frac{a^3b^2 - ab^2}{6a}$, clearly using a "cancelling" model. Other common errors included evaluating $\log_2 14$ given $\log_2 7 = 2.807$ as $\log_2 14 = 2\log_2 7$, apparently thinking that $\log_a xy = x\log_a y$.

This way of conceptualising skills with logarithms provides a useful framework for thinking about how students *could* think about logarithms. It does not give us detailed information about how they actually *do* operate with logs and the way in which they construct algebraic understandings in a logarithmic context. This study was designed to help provide this kind of information. The details of the study and its results follow in the next two sections.

Method

This research project involved the gathering of data through a detailed test instrument which was administered to students in three schools (two were used for the main study while the third was used to pilot the test instrument only).

The Participants

The three schools in the study, labelled A, B and C, are all mixed-gender, government schools. School A (the pilot study school) has a student population of about 1100, is located in the eastern part of Singapore, and caters to the needs of a wide range of students from Secondary 1 to 5. School B, also located in the eastern part of Singapore, has a population of about 1440 with an academically talented student base, all of whom are in the Express Stream. School C opened in the year 2000 in the western part of Singapore. The student population in the three streams has grown from about 150 in the year 2000 to about 1000 in the year 2003 with 27 classes from Secondary 1 to 4. In terms of the students' Primary School Leaving Examination (PSLE) aggregate scores, the cohort of Express students in School C is comparable to School A.

In total, 81 Secondary Three Express students, 42 from School B and 39 from School C, sat for the paper-and-pencil test in the main study. Those from School B, comprising 25 male and 17 female, were randomly selected from six classes taught by different additional mathematics teachers whereas those from School C, comprising 17 male and 22 female, were mainly from the same class except for eight of them coming from another class. The PSLE aggregate scores of the 81 students varied from a low of 191 to a high of 246, suggesting a wide range of learning abilities.

Test Instrument

As no test instrument specifically to explore students' learning difficulties, misconceptions and errors in logarithms was available, a new test instrument had to be developed to achieve the aims of this study. The Test of Students' Understanding of Logarithms (ToSUL) is a paper-and-pencil test with 47 items. In devising the test instrument, an attempt was made to cover a wide range of typical test items found in the mathematics textbooks used in the schools although the test items did not involve natural logarithms. Since one of the purposes of this study was to examine both skills and conceptual understandings of logarithms, considerable effort went into ensuring that the items represented a range of cognitive demand. A modified version of Bloom's Taxonomy (Bloom et al., 1956) was used to sort the items into three categories: Computation or Knowledge, Understanding and Application.

The sorted items were given to three experienced colleagues who also classified the items. Some suggestions for improvements of the test items, as well as the classification criteria for each category, were provided by them and the test instrument was revised based on their feedback. At this point at least two out of three agreed on the classification of 90% of the items. Samples of items in each category are provided in Figure 1 to give the reader a better sense of the kinds of things that were being tested.

The *Knowledge or Computation* category comprised routine questions that require direct recall or application of the definition and laws of logarithms, as well as simple manipulation or computation with answers obtained within two to three steps. Of the 47 test items in the main test, 23 items belong to this category. The items in the *Understanding* category do not just simply involve recalling the definition or the application of logarithmic laws, but require some understanding of the underlying concepts of logarithms. The items may be familiar or textbook-like, but in them the student must decide not only what to do but how to do it. There were 14 test items in this category. Finally, items in the *Application* grouping

Category	**Item**
Knowledge or Computation	Given that $a^m = 36$, find $\log_a 36$, in terms of *m*.
Understanding	Given that $a^m = 36$, find $\log_6 a$, in terms of *m*.
Application	Find the greatest possible integral value of *p* and the least possible integral value of *q* given that $p < \log_{10} 500 < q$.

Figure 1. Sample Test Items by Category

require the students to develop their own techniques for solving problems that they probably have not met in a textbook. There are 10 application items in the main test.

This instrument was piloted in school A with 43 students. Based on the analysis of students' responses, the queries some students had, and observations made during the invigilation of the pilot test, a few items were clarified or rewritten. The time allocation of 90 minutes seemed slightly long and was revised to 75 minutes.

Data Analysis

The revised test was administered (without the use of calculators) to 81 students from Schools B and C on two separate days in July 2003. The resulting scripts were collected but because two students were unwell their scripts were rejected, leaving 79 scripts for data analysis. Before marking them, each script, in the order they were collected, was coded from M1 to M79. Once the marking was completed, a detailed item-by-item analysis was carried out by examining the participants' responses for each test item using four categories: correct answer, incorrect answer, incomplete solution or blank solution. Unlike the categories of "correct answer", "incomplete solution" and "blank solution", the responses under the "incorrect answer" category were more varied and had to be further analyzed into distinct categories based on some common features. The "incomplete solution" category was created originally to account for those students whose solutions were incomplete but could possibly lead to a correct answer if done fully. However, due to a low occurrence of such responses for each item, it was decided to merge this category with the "did not attempt" category, which accounted for those who left their solutions completely blank. Hence the categories of responses were narrowed from four to three. Lastly, the frequency of response for each category in a test item was computed.

To illustrate, consider Item 6(b) (see Table 1). The correct answer is indicated with an asterisk. All incorrect answers were further classified according to some common features that matched one of the following five categories: *numerical values*, *in terms of m, in terms of a, in terms of logarithms* or *evaluating m or a*. Within each category, every frequently occurring incorrect answer was represented separately while the remaining incorrect answers were collectively grouped under "Others". For instance, the response of $\frac{m}{2}$ was regarded as a frequently occurring incorrect answer within the "in terms of *m*" category because of the 7 occurrences, high in comparison with the occurrences of other responses such as

Table 1
Item Analysis for Test Item 6(b)

Item 6b: Given that $a^m = 36$, find $\log_6 a$ in terms of *m*.	**Level:** U
Responses	**Frequency**
$\frac{2}{m}$ *	27
Numerical values	
2	2
Others	4
In terms of *m*	
$\frac{m}{2}$	7
Others	11
In terms of *a*	2
In terms of logarithms	2
Evaluating *m* or *a*	
$a = 6^m$	4
$m = 2$	1
Did not attempt / attempted but working is incomplete	17
$m = 2\log_a 6$	2

m, $2m$, $6m$, $\frac{1}{m}$, $\frac{1}{2m}$ and $\frac{1}{\sqrt{m}}$ that were grouped under "Others". The response of $m = 2\log_a 6$ was seen twice, but it was treated as an incomplete solution for two reasons. First, it would be unjustifiable to classify it as an incorrect answer when this partially correct response could possibly lead to the correct answer if the participant had proceeded further as follows:

$$m = 2\log_a 6$$

$$\text{So } \log_a 6 = \frac{m}{2}$$

$$\text{Since } \log_6 a = \frac{1}{\log_a 6},\ \log_6 a = \frac{2}{m}$$

Second, it was shown in the working space provided and not in the answer space, so it was uncertain whether or not $m = 2\log_a 6$ was the intended final response. Hence this response was included under the "Did not attempt / attempted but working is incomplete" category.

In addition, all incorrect answers found were analysed carefully to determine the potential causes of the errors. The causes were categorised and then grouped so that eventually the number of distinct categories was collapsed to three: Errors due to deficient mastery of concepts, rules and pre-requisite skills; errors due to over-generalisation; and, miscellaneous (indecipherable errors, slips due to incorrect arithmetic computations, incorrect algebraic manipulations, errors due to guessing, and bad handwriting).

Given the establishment of these error types, all incorrect answers were then re-analysed and coded accordingly. A few days later, they were coded again to ensure consistency in the coding process.

Findings

The results of this analysis yielded a rich source of information about students' skills and knowledge when working with logarithms. Here we present both general remarks about the participants' level of understanding of logarithms by examining their performance in the three categories of test items and a more detailed description of some of their common errors and the possible misconceptions behind them.

<u>Overall Level of Understanding of Logarithms</u>

Table 2 summarises the performance outcomes of 79 participants in the ToSUL test. The *Knowledge* or *Computation* category is the lowest cognitive level of the three while the *Application* category is the highest.

Table 2
Performance Outcomes in ToSUL

Cognitive Level	Number of Test Items	Maximum Possible Number of Correct Responses	Number of Correct Responses	Percentage of Correct Responses
Knowledge or Computation (K/C)	23	1817	1553	86%
Understanding (U)	14	1106	729	66%
Application (A)	10	790	304	39%

The table clearly indicates that the participants were very successful in answering *Knowledge* or *Computation* test items with a success rate of about 86%, but, predictably, when the items became more difficult, the percentages dipped to a level of about 66% for *Understanding* items and to a low of about 39% for *Application* items. These findings suggest that most participants were successful with routine and familiar items, however, they did not perform as well when the items deviated slightly from the familiar format or involved some applications of logarithms. These results are not surprising and are, in fact, consistent with Skemp's (1976) ideas about instrumental and relational understanding and the relative difficulty of achieving each.

Knowledge/Computation Items

In this grouping, there were, however, a few questions with very poor success rates. For example, when asked for the value of $\log_t\left(\frac{1}{t}\right)$, the response of $\log_t 1 - 1$ suggested that some participants did not appreciate what it means to find the value of an expression. In this case, the participants did not proceed further to evaluate $\log_t 1$ as zero and hence to simplify the expression to -1. Surprisingly, when asked to evaluate $\lg 100$, 28% of the 79 participants gave the response of 10 rather than the expected 2. It appears that even an instrumental understanding of logarithms is difficult for some students to acquire.

Understanding Items

The overall success rate in this category was approximately 66%, suggesting that the participants had demonstrated a reasonable understanding of the underlying concepts of logarithms beyond simply recalling the definition or the direct application of logarithmic laws; however, none of the 14 test items in this category had more than 70 participants giving correct answers. For example, although 69 participants gave the correct answer 4 as the solution for the equation $\log_w 16 = 2$, only 27 participants provided a correct reason why it was the only solution. When asked to express $\log_6 m$ in terms of *m* given $a^m = 36$, there were only 27 correct answers and about 25% of the participants did not attempt the question. In an item which asked for the simplification of $\frac{\log_2 27}{\log_2 9}$, responses of 3 (about 14%) and $\log_2 3$ (about 23%) were all relatively common. Both responses appear to reveal some misconceptions. The first arises probably by treating $\log_2$ as a variable whereas the latter arises possibly from participants thinking $\log x \div \log y = \log(x \div y)$.

Application Items

This category comprises a number of more challenging test items which required students to develop their own techniques for solving a problem which they probably had not encountered before. A significant number of the participants did appear to have some difficulty with the items, with half of them not well answered. The overall success rate for this category was approximately 39%, varying from a high of 67% to a low of 6%. A typical question asked participants to express $\log_3 v$ and $\log_v 81$ respectively in terms of *u* when $u = \log_9 v$, but it appeared that the majority of participants could not envisage the connection between the terms and the given condition. Another example of a lack of depth in their understanding of logarithms was demonstrated in an item which asked them to find, with justification, the possible value(s) of *x* satisfying the equation $\log_2(x+1) + \log_2(x-2) = \log_2(3x-5)$. They regularly found the two solutions of $(x+1)(x-2) = (3x-5)$, but as many as 31 participants failed to reject one of the two solutions by checking their validity. Of those who checked, there were some who did not recognise that the response of 1, though a positive value, was inadmissible because two of the terms in the equation, $\log_2(x-2)$ and $\log_2(3x-5)$, were undefined when $x = 1$. The item with the lowest correct response rate (6%) asked them to evaluate the expression $2^{\log_2 5}$. What makes this interesting is that in spite of evidence showing high success rates in the conversion of logarithmic equations to their index form and vice-versa, students were not able to extend this understanding to a non-standard case.

This summary analysis confirmed what we as teachers have typically suspected – students can often do the mechanical things we ask with logarithms but there is not always a high level of understanding behind these computations. Examining the specific details of errors in order to try to establish possible reasons for them was the next stage of our analysis. The following section discusses in more detail the system for categorising errors that was employed and suggests some plausible explanations for the kinds of errors that typify these categories. Based on this analysis, we conclude with a series of suggestions for teachers to help prevent students from developing these kinds of misconceptions.

Discussion

As detailed above, the various errors were sorted into three categories. The most significant were errors due to over-generalisation (OG) of concepts and rules, followed by errors due to a deficient mastery of concepts, rules and pre-requisite skills (D). Despite the fact that the miscellaneous (M) category encompasses quite a few different kinds of errors, it was the least common type. Table 3 presents the number of incorrect answers in each category, together with the number of blank or incomplete solutions.

Table 3
Classification of Unsuccessful Responses

	Incorrect answers			**Blank or Incomplete Solutions**
Response Codes	**OG**	**D**	**M**	**X**
Number of Responses	369	318	196	241
Percentages	33%	28%	17%	22%

As revealed in the table, the high occurrence of errors due to over-generalisation suggests that when errors are made, they are not, as is often assumed, due to carelessness or insufficient practice on the part of students. Rather it appears that these errors are due to misconceptions that students have actively constructed when they use their existing schema to interpret new ideas. Therefore, unlike errors due to a deficient mastery of concepts, rules and pre-requisite skills which can be overcome by more practice, these misconceptions are grounded in faulty understanding and consequently cannot be addressed by more drill and practice of the standard type. This is not to suggest that students could not benefit from more practice but rather to emphasise that the *kind* of practice is just as significant as the quantity.

Consider, for example, errors that are categorised as being due to over-generalisation. Consistent with the findings by Kaur and Boey (1994) and Yen (1999), the typical mistake of assuming $\log_a$ is a variable rather than an operation was very much in evidence in this study. A typical manifestation occurred when students used the change-of-base law to rewrite $\lg 100$ as $\frac{\lg 100}{\lg 10}$ and then simplify it by cancelling lg from both the numerator and the denominator, resulting in the response of 10. This mistake of cancelling $\log_a$ or lg from both the numerator and the denominator is probably generalised from the algebraic rule that $\frac{wx}{wy}$ can be simplified to $\frac{x}{y}$ by the cancellation of the variable *w* from the numerator and the denominator.

Other than this type of error, the conceptualisation of $\log_a$ as a variable also led to other errors, again possibly as a result of over-generalisation of rules. For instance, it was quite common to find participants treating $\log_a$ as a common factor and factorising $\log_a x + \log_a y$ into $\log_a(x+y)$, just like factorising $2x+2y$ into $2(x+y)$. For instance, some participants simplified $\log_x 6 - 2\log_x 3$ correctly to $\log_x 6 - \log_x 9$, but, surprisingly, they continued by treating $\log_x$ as a common factor and obtained a term, $\log_x(-3)$, that is undefined. Additionally, it was noted that this term was further simplified to $-\log_x 3$!

Another quite common misconception arose from participants thinking that $\log x \div \log y = \log(x \div y)$. This misconception is well illustrated by the typical (but faulty) solution: $\dfrac{\log_2 27}{\log_2 9} = \log_2 27 \div \log_2 9 = \log_2(27 \div 9) = \log_2 3$. Many more examples could be given but the principle is essentially the same. Students take new knowledge and try to make sense of it using previously learned schema. Sometimes they are successful, and sometimes they are not, but when they develop misconceptions they are not without some sort of logic. The roots of such flawed understandings are impossible to establish as they are likely a combination of factors: the student's own ideas, imprecise statements made by the teacher, friends' explanations that are incomplete, and so on. Finding the cause, however, is no less important than a realisation that such misconceptions are common and instruction must specifically take the likelihood of their occurrence into account.

Implications for Instruction

The results of this study point to certain kinds of errors as being quite common, notably those where students fail to see the logarithm as a number. It is this failure that then leads them to pull expressions with logarithms apart and do things such as cancel the word "log". However, when one chooses to introduce the topic, we all want students to be able to evaluate and understand the meaning of expressions like $\log_{10} 1000$ or $\log_5 25$. We think it is useful to take some time at the start and have students first read the logarithmic expression and then explain its meaning before trying to evaluate it. So, for example, if they were presented with $\log_2 32$, have them read it as "log to the base 2 of 32" and then give its meaning as "the exponent required on the base 2 to produce the value 32". Then they can easily evaluate the expression to give 5. This is an attempt to get them to see that the expression $\log_2 32$ is not three separate entities but rather a single numerical value which is in fact an exponent. There are very few logarithmic expressions that secondary students encounter that cannot be evaluated by going back to this basic and fundamental understanding.

Consider the expression $2^{\log_2 5}$. A typical algebraic approach would be to let $x = \log_2 5$ and then rewrite this equation in exponential form as $2^x = 5$. Substituting for *x* yields the fact that $2^{\log_2 5} = 5$. There is nothing mathematically wrong with this approach; however, there is a tendency on students' parts to simply memorise how to translate between logarithmic and exponential form just in terms of where the place holders are. In other words when faced with an expression in logarithmic form they put the small number as the base, the number after the log as the result and then the number on the other side of the equal sign as the exponent. This piece-meal re-arrangement further reinforces the incorrect idea that somehow the expression $\log_2 5$ is made up of separate pieces. Recall also that this question was done correctly by only 6% of the respondents.

Despite the initial difficulty, it may be preferable to have students evaluate the expression $2^{\log_2 5}$ by going back to fundamentals. Ask them to explain the meaning of $\log_2 5$. It is "the exponent required on the base 2 to produce the result 5". Consequently if we put this exponent on 2 we should not be surprised to get 5!

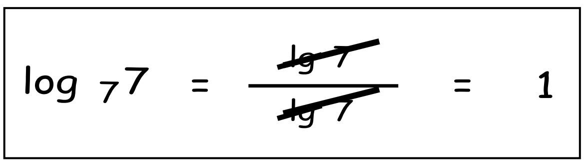
New notation is always difficult for students and the $\log_a b$ notation is especially so because it bears a resemblance to algebraic notations with which they are familiar. It is not surprising that students commonly misconstrue the notation as a variable, rather than an operator. For instance, $\log_a bc$ is quite often mistaken as the product of $\log_a b$ and *c*, which appears to resemble the associative law of multiplication if $\log_a$, b and c are treated as variables. So, it is important for teachers to make clear at the beginning of the topic what the notation $\log_a$ actually represents. They can explain and emphasise to students that the notation in the term $\log_a b$ is actually an instruction (or an operator) to find the exponent required on the base *a* to give the value *b*, and that $\log_a b$, when treated as a whole, is simply the exponent to be found. For instance, in the case of $\log_a bc$, $\log_a$ is an operator on bc to find the exponent required on the base *a* to give the value bc. Alternatively, the use of brackets to distinguish clearly the difference between $\log_a(bc)$ and $(\log_a b)c$, followed by an interactive discussion with students to talk about the difference, in the initial stage of learning may also be helpful. Regardless of the strategy used, students should know how to interpret logarithmic notations correctly. For instance, it should be clear to them that $\log_a bc$ is taken to mean $\log_a(bc)$, and not $(\log_a b)c$.

If teachers are aware of these common errors and their possible causes, they can actually use them as opportunities for learning rather than see them as inevitable problems. For example, knowing that students may be tempted to think that $\log_a x + \log_a y = \log_a (x + y)$, teachers can engage students in a discussion of the answer to the question $\log_6 2 + \log_6 3$, to decide whether or not the answer is $\log_6 5$. Teachers can highlight why the answer is wrong by getting students to verify the values of $\log_6 2 + \log_6 3$ and $\log_6 5$ by using calculators. With a few more similar illustrations, students can be convinced that although for some values of *x* and *y* it is true that $\log_a x + \log_a y = \log_a (x + y)$, it is not true in general.

Errors can also be used as "springboards for inquiry" (Borasi, 1987, 1994) to address misconceptions during teaching. Teachers can involve students in activities organised around the explicit study of some previously selected errors or impromptu errors made by the students during the lessons. For instance, a worksheet containing both correct and incorrect solutions to some questions on logarithms can be given to students to directly engage them in the error analysis and, at the same time, to encourage them to pursue open-ended explorations and reflections.

Finally, errors can be used as springboards for cognitive conflicts to provoke students' thinking and to guide them to correct their errors themselves. For instance, when students obtain $\log_{10} 500 = 50$ by thinking that $\log_{10} 500$ is fifty times as much as $\log_{10} 10$, they can be asked to apply the same method to evaluate, for instance, $\log_{10} 100$ just to see if the resulting answer is what they are expecting. The resulting answer will probably lead to a conflicting outcome if the misconception is present. For this strategy to work effectively, it is important to provide students with immediate feedback so that errors and misconceptions are challenged as soon as they occur. Similarly, it is just as important to engage them in an interactive discussion to talk about their work, and perhaps to justify their answers as well.

Teachers also need to be mindful of how they say things so as not to leave the wrong impression in a student's mind. For instance, the use of phrases such as "times means add" and "divide means subtract" when helping students to remember the product rule and quotient rule of logarithms can be both misleading and unhelpful, and contribute to the development of quick, but faulty, rules by the student. Careful written work at the whiteboard is also important. For example, consider the demonstration of finding the value of $\log_7 7$ by using the change-of-



Correct Cancellation

base rule. After expressing $\log_7 7$ as $\frac{\lg 7}{\lg 7}$, it is important that teachers pay special attention to how the simplification of $\frac{\lg 7}{\lg 7}$ is being written. For instance, the short strokes, used to denote the cancellation of the numerator and the denominator, have to unambiguously cross out the entire $\lg 7$ in both the numerator and the denominator because if done in a way that only shows a crossing out of the notation lg , then the answer of 1 may be mistaken as having been obtained by cancelling lg indiscriminately from both the numerator and the denominator, followed by the division of 7 by itself and this action tends to inadvertently reinforce precisely the misconception that students tend to develop.

Conclusion

It appears that although most students seem to have acquired an instrumental understanding of logarithms they still make many errors due to over-generalisation of rules learned previously. As a result, this study supports both the common impression that the topic of logarithms is a difficult one for students and that there is a high prevalence of misconceptions in students' thinking. It is hoped that with a greater awareness of these difficulties, teachers can plan more effective teaching and learning experiences that recognise and anticipate the potential misconceptions that may arise in their students' thinking.

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